Math Analysis II
HW 1
Due 01/31/2013

1.

a) Explain the relationships between metric, normed linear, and inner-product spaces

b) Explain the following concepts and give an example illustrating the importance of each.

1) Triangle Inequality
2) Cauchy-Schwartz inequality
3) Parallelogram identity

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2.

a) Let M be a set. Show that the function $d(x, y)$ defined as

$$d_{\text{dis}}(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

is a metric on M. (This metric is called the discrete metric.)

b) Let $S(x_0; 1) = \{x \in M : d(x_0, x) = 1\}$ be the sphere centered at x_0 with radius 1 and $B_c(x_0; r)$, $B(x_0; r)$ denote the closed and open ball center at x_0 with radius r respectively. Describe $S(x_0; 1)$, $B_c(x_0; r)$ and $B(x_0; r)$ in the metric space (M, d_{dis}).

c) Let M be an infinite set with a discrete metric. Show that (M, d_{dis}) is closed and bounded but not compact.

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3. Show that for a set $B_c(x, 1) = \{x \in R^n : ||x|| \leq 1\}$ is compact and convex.

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4. Let $A \subset \mathbb{R}^n$ be compact and let (x_k) be a Cauchy sequence in $\mathbb{R}^n$ with $x_k \in A$. Show that x_k converges to a point in A.

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5. Let X be a complete metric space and (F_k) be a sequence of closed nonempty subsets satisfying the property that $F_{k+1} \subset F_k$ for all $k = 1, 2, \ldots$ (i.e. the nested set property) such that $\mathrm{diam}(F_k) \to 0$ as $k \to \infty$. Show that there is exactly one point in $\bigcap F_k = F$. (Recall that diameter of F_k, $\mathrm{diam}(F_k) = \sup\{d(x, y) : x, y \in F_k\}$.)

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6. Let K be a compact subset of (M, d) and $f : K \to \mathbb{R}$ is a continuous function then prove that

a) $f(K)$ is compact in $\mathbb{R}$.

b) There exists points $x_0, x_1 \in K$ such that $f(x_0) = \inf(S)$ and $f(x_1) = \sup(S)$ where $S = \{f(x) : x \in K\} \subset \mathbb{R}$.

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7. Show that for a subset $A \subset (M, d)$ we have $x \in \overline{A} \iff$ there exists a sequence $(x_k) \in A$ such that $x_k \to x$ where by $\overline{A}$ we mean the closure of A.

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8. A subset K of a metric space M is called <u>dense</u> if $\overline{K} = M$. M is called separable if M contains a countable dense subset. Show that $\mathbb{R}$ and $\mathbb{R}^n$ are separable metric spaces.

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9. Prove that a totally bounded metric space (M, d) is separable.

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10. The set measure of noncompactness $\alpha(D)$ for a bounded subset D of (M, d) is defined as

$$\alpha(D) = \inf\{r > 0 : D \subset \bigcup_{i=1}^{n} A_i \ \mathrm{diam}(A_i) \leq r\}.$$

Show that

a) if D is compact then $\alpha(D) = 0$.

b) if $D_1 \subset D_2 \implies \alpha(D_1) \leq \alpha(D_2)$ (α is monotone)

c) $\alpha(\overline{D}) = \alpha(D)$ (invariant when given the closure)

d) If $\{F_n\}$ is a decreasing sequence of non-empty closed and bounded subsets of a complete metric space (X, d) and if $\lim_{n\to\infty} \alpha(F_n) = 0$ then the intersection of all the F_n is non-empty and compact. (This is called the Generalized Cantor's intersection theorem.)

11. Respond to the following

a) Define the Cantor Set C

b) Show that the Cantor Set is compact

c) Show that the length of C is equal to 0

d) Show that the Cantor Set can be put into a one-to-one correspondence with the interval $[0, 1]$

e) Show that $\text{Card}(C) = \mathbf{c}$, where $\mathbf{c}$ is the cardinality of the real line